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Equivalence Postulate and Quantum Origin of Gravitation

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We suggest that quantum mechanics and gravity are intimately related. In particular, we investigate the quantum Hamilton–Jacobi equation in the case of two free particles and show that the quantum potential, which is attractive, may generate the gravitational potential. The investigation, related to the formulation of quantum mechanics based on the equivalence postulate, is based on the analysis of the reduced action. A consequence of this approach is that the quantum potential is always non–trivial even in the case of the free particle. It plays the role of intrinsic energy and may in fact be at the origin of fundamental interactions. We pursue this idea, by making a preliminary investigation of whether there exists a set of solutions for which the quantum potential can be expressed with a gravitational potential leading term which alone would remain in the limit $\hbar \to 0$. A number of questions are raised for further investigation.

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1 Introduction

According to the recently formulated Equivalence Principle (EP), all physical systems are equivalent under coordinate transformations 1.–5. It has been shown that the implementation of such a principle unequivocally leads to the Quantum Hamilton–Jacobi Equation (QHJE). The latter was first analyzed independently by Floyd in a series of remarkable papers 6. In 5. the formulation of Quantum Mechanics (QM) from the EP was extended to higher dimensions and to the relativistic case as well. This approach suggests that QM and General Relativity (GR) are two facets of the same medal 1.–5. In this letter we will argue that QM and GR are intimately related. In particular, we suggest that gravitation has a purely quantum mechanical origin.

An outcome of the formulation of QM based on the EP is that the term $\mathcal{W} \equiv V - E$, with V the potential and E the energy of the system, corresponds to the inhomogeneous term in the transformation properties of the state with $\mathcal{W} = \mathcal{W}^0 \equiv 0$ (see Refs. 1.–5.). It turns out that this term is of a purely quantum nature. A related aspect concerns the appearance of fundamental constants in the QHJE. In particular, the implementation of the EP leads to the introduction of universal length scales. This has an important consequence once we take into account that the quantum potential is always non–trivial. This is a result which follows from a rigorous analysis of the QHJE. Here and throughout this paper it is important to distinguish between the quantum potential arising in the approach adopted here and that in the Bohm theory of QM 7. 8. The two are not in general the same (see Refs. 6. 9. 1.-5.). In particular, it turns out that even in the case of $\mathcal{W}^0$, the corresponding quantum potential is far from being trivial. This key point is due to the fact that the quantum reduced action, or quantum Hamiltonian characteristic function, is always non–trivial. In particular we have

$$\mathcal{S}_0 \neq cnst, \tag{1}$$

which follows as a direct consequence of the EP 1.–5.

2 The equivalence postulate

Before proceeding, let us analyze further the EP. To understand the basic motivation for its formulation, let us consider, in the classical framework, two particles of mass m_A and m_B with relative velocity v 4. For an observer at rest with respect to the particle A, the two systems have reduced actions $\mathcal{S}_0^{cl\,A}(q_A) = cnst$ and $\mathcal{S}_0^{cl\,B}(q_B) = m_B v q_B$. In this case, setting $\mathcal{S}_0^{cl\,A}(q_A) = \mathcal{S}_0^{cl\,B}(q_B)$ defines a highly singular coordinate transformation. However, when the same system is described by an observer in a frame which is not at rest with respect to either A and B, we have that equating the two reduced actions does not lead to such a singularity. Thus, this strong singularity disappears if the frame one uses to describe the systems A and B has a non–zero velocity with respect to both. For example, if the observer is in a frame moving with constant acceleration a with respect to the systems A and B, then

$$\tilde{\mathcal{S}}_0^{cl\,A}(Q_A) = \frac{m_A}{3a}(2aQ_A)^{\frac{3}{2}}, \qquad \tilde{\mathcal{S}}_0^{cl\,B}(Q_B) = \frac{m_B}{3a}(v^2 + 2aQ_B)^{\frac{3}{2}}, \tag{2}$$

where Q_A (Q_B) is the coordinate of particle A (B) in the accelerated frame. If in describing particle B in the accelerated frame one uses the coordinate Q_A defined by $\tilde{\mathcal{S}}_0^{cl\,A}(Q_A) = \tilde{\mathcal{S}}_0^{cl\,B}(Q_B)$, then the resulting dynamics coincides with the one of particle A, that is

$$\tilde{\mathcal{S}}_0^{cl\,B}(Q_B(Q_A)) = \tilde{\mathcal{S}}_0^{cl\,A}(Q_A), \tag{3}$$

which shows that the system B, described in terms of the coordinate Q_A, coincides with the system A. Hence, in Classical Mechanics (CM), the equivalence under coordinate transformations requires choosing a frame in which no particle is at rest. The existence of a distinguished frame, the one at rest, seems peculiar as on general grounds what is equivalent under coordinate transformations in all frames should remain so even in the one at rest. This leads to postulate that it is always possible to connect two systems by a coordinate transformation. In other words, it is natural to require that given two systems with reduced actions $\mathcal{S}_0$ and $\mathcal{S}_0^v$, there always exists the "v–map" $q \rightarrow q^v$ defined by 1.–5.

$$\mathcal{S}_0^v(q^v) = \mathcal{S}_0(q). \tag{4}$$

The above example shows that the HJ formalism provides the natural setting to describe physical systems. The equivalence under coordinate transformations is somehow in the spirit of general relativity. This property of HJ theory is still more evident if one notes that the classical HJ equation itself is obtained by looking for the canonical transformation of the conjugate variables that leads to the trivial Hamiltonian $H = 0$. Thus, according to CM, since all states are equivalent, in the sense of the canonical transformations, to the trivial one, there is a sort of EP. Our view is slightly different from the one considered in the framework of canonical transformations. Actually, it is just the above example, in which equivalence under coordinate transformations always exist except that in the case one considers the particle at rest, which suggests the following stronger concept of equivalence. Let us denote by $\mathcal{H}$ the space of all possible states $\mathcal{W}$. The equivalence postulate reads 1.–5.

For each pair $\mathcal{W}^a, \mathcal{W}^b \in \mathcal{H}$, there exists a v–transformation such that

$$\mathcal{W}^a(q) \longrightarrow \mathcal{W}^{av}(q^v) = \mathcal{W}^b(q^v). \tag{5}$$

It has been shown in 1.–5. that the implementation of the EP unequivocally leads to the quantum HJ equation in any dimension and in the relativistic case as well.

3 Fundamental constants and the quantum potential

Due to the structure of the QHJE we have that the quantum potential will in general depend on fundamental constants. Let us show how these constants arise. We first focus on the one–dimensional Quantum Stationary Hamilton–Jacobi Equation (QSHJE). This reads

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0(q)}{\partial q}\right)^2 + V(q) - E + \frac{\hbar^2}{4m}\{\mathcal{S}_0, q\} = 0, \tag{6}$$

where $\{\mathcal{S}_0, q\} = \mathcal{S}_0'''/\mathcal{S}_0' - 3(\mathcal{S}_0''/\mathcal{S}_0')^2/2$ denotes the Schwarzian derivative and $Q = \frac{\hbar^2}{4m}\{\mathcal{S}_0, q\}$ is the quantum potential. The general real solution of (6) has the form

$$e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta\}} = e^{i\alpha}\frac{w + i\bar{\ell}}{w - i\ell}, \tag{7}$$

where $w = \psi^D/\psi \in \mathbb{R}$ and (ψ^D, ψ) are two real linearly independent solutions of the associated Schrödinger equation. Furthermore, we have $\delta = \{\alpha, \ell\}$, with $\alpha \in \mathbb{R}$ and $\ell = \ell_1 + i\ell_2$ some integration constants ($\bar{\ell}$ denoting the complex conjugate of ℓ). Observe that $\ell_1 \neq 0$, which is equivalent to having $\mathcal{S}_0 \neq cnst$, is a necessary condition to define the term $\{\mathcal{S}_0, q\}$ in the QSHJE.

There is a simple reason why fundamental constants should be hidden in ℓ. To see this, consider the Schrödinger equation in the trivial case $\mathcal{W}^0(q^0) \equiv 0$, that is $\partial^2_{q^0}\psi = 0$. Two linearly independent solutions are $\psi^D = q^0$ and $\psi = 1$. Now a basic aspect of the formulation is manifest duality between real pairs of linearly independent solutions 1.–5. This is a fact which is strictly related to the Legendre duality first observed in 10. and further investigated in 11.–14. Thus, whereas in the standard approach one usually considers only one solution of the Schrödinger equation, *i.e.* the wave–function itself, in the present formulation both ψ^D and ψ enter the relevant formulas. This leads to expressions containing linear combinations of ψ^D and ψ, typically $\psi^D + i\ell\psi$ that for $\psi^D = q^0$ and $\psi = 1$ reads $q^0 + i\ell_0$, so $\ell_0 \equiv \ell$ should have the dimensions of a length. The fact that ℓ has the dimensions of a length is true for any state. This follows from the observation that the ratio $w = \psi^D/\psi$ is a Möbius transformation of the trivializing map transforming any state to $\mathcal{W}^0$ 1.–5. Hence w, and therefore ℓ, has the dimensions of a length.

Since ℓ_0 enters the QSHJE with $\mathcal{W}^0 \equiv 0$, the system does not provide any dimensionful quantity. This implies that we have to introduce some fundamental lengths. Let us show this in some detail. The reduced action $\mathcal{S}_0^0$ corresponding to the state $\mathcal{W}^0$ is

$$e^{\frac{2i}{\hbar}\mathcal{S}_0^0\{\delta\}} = e^{i\alpha}\frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}, \tag{8}$$

and the conjugate momentum $p_0 = \partial_{q^0}\mathcal{S}_0^0$ has the form

$$p_0 = \pm\frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}. \tag{9}$$

A property of p_0 is that it vanishes only for $q^0 \to \pm\infty$. Furthermore, $|p_0|$ reaches its maximum

at $q^0 = -\mathrm{Im}\,\ell_0$

$$|p_0(-\mathrm{Im}\,\ell_0)| = \frac{\hbar}{\mathrm{Re}\,\ell_0}. \tag{10}$$

Since $\mathrm{Re}\,\ell_0 \neq 0$, p_0 is always finite. Thus, $\mathrm{Re}\,\ell_0 \neq 0$ provides a sort of ultraviolet cutoff. This is a property which extends to arbitrary states. Actually, the conjugate momentum has the form

$$p = \frac{\hbar W(\ell + \bar{\ell})}{2\,|\psi^D - i\ell\psi|^2}, \tag{11}$$

where $W = \psi'\psi^D - \psi^{D\prime}\psi$ is the Wronskian. Since W is a non–vanishing constant, it follows that ψ^D and ψ cannot have common zeroes, and by $\mathrm{Re}\,\ell \neq 0$ we see that p is finite $\forall q \in \mathbb{R}$. Therefore, the EP implies an ultraviolet cutoff on the conjugate momentum.

In Refs. 2. and 4. it has been shown that fundamental constants also arise in considering the classical limit. In particular, one first considers

$$\lim_{\hbar\to 0} p_0 = 0, \tag{12}$$

and note that $\mathrm{Im}\,\ell_0$ in (9) can be absorbed by a shift of q^0. Hence, in (12) we can set $\mathrm{Im}\,\ell_0 = 0$ and distinguish the cases $q^0 \neq 0$ and $q^0 = 0$. From (12)

$$p_0 \underset{\hbar\to 0}{\sim} \begin{cases} \hbar^{\gamma+1}, & q_0 \neq 0, \\ \hbar^{1-\gamma}, & q_0 = 0, \end{cases} \tag{13}$$

where $-1 < \gamma < 1$ with γ defined by $\mathrm{Re}\,\ell_0 \underset{\hbar\to 0}{\sim} \hbar^\gamma$. The are not many fundamental lengths in nature. In particular, we note that a fundamental length satisfying this condition on the power of $\hbar$ is the Planck length $\lambda_p = \sqrt{\hbar G/c^3}$, while the Compton length is excluded by the condition $\gamma < 1$. Also, as we will see in considering the $E \to 0$ and $\hbar \to 0$ limits for the free particle of energy E, the natural choice is just the Planck length. With this choice of $\mathrm{Re}\,\ell_0$ the maximum of $|p_0|$ is

$$|p_0(-\mathrm{Im}\,\ell_0)| = \sqrt{\frac{c^3\hbar}{G}}. \tag{14}$$

Setting $\mathrm{Im}\,\ell_0 = 0$ and $\mathrm{Re}\,\ell_0 = \lambda_p$, the quantum potential associated to the trivial state $\mathcal{W}^0$ is

$$Q^0 = \frac{\hbar}{4m}\{\mathcal{S}_0^0, q^0\} = -\frac{\hbar^3 G}{2mc^3}\frac{1}{|q^0 - i\lambda_p|^4}. \tag{15}$$

There are two basic aspects in this expression. Firstly, the gravitational constant G results from ensuring consistency with the classical limit. We saw that this arises naturally as a consistency condition. Furthermore, Q^0 is negative definite. Thus, even if we are still considering a one–dimensional system, we are starting to see some motivation for the emergence of the gravitational interaction. In particular, note that this analysis is essentially the same of the one in three dimensional space as in the case of a free particle we can consider a reduced action of the form $\mathcal{S}_0(x)+\mathcal{S}_0(y)+\mathcal{S}_0(z)$, which can always be chosen as a possible solution of the QHSJE when the potential has the form $V(x,y,z)=V_1(x)+V_2(y)+V_3(z)$. We also note that the fact that we are in the framework of non–relativistic QM, does not exclude the appearance of c in the integration constants of the QSHJE (an example of the appearance of c in QM is the non–relativistic treatment of an electron in a magnetic field).

The appearance of fundamental constants can also be seen by considering the $\hbar\longrightarrow 0$ and $E\longrightarrow 0$ limits 2. for the conjugate momentum of a free particle of energy E

$$p_E=\pm\frac{\hbar(\ell_E+\bar{\ell}_E)}{2\left|k^{-1}\sin(kq)-i\ell_E\cos(kq)\right|^2}, \tag{16}$$

where $k=\sqrt{2mE}/\hbar$ and ℓ_E is the integration constant of the QSHJE. We should require that (see Refs. 2. and 4.)

$$\lim_{\hbar\to 0}p_E=\pm\sqrt{2mE}, \tag{17}$$

and

$$\lim_{E\to 0}p_E=p_0=\pm\frac{\hbar(\ell_0+\bar{\ell}_0)}{2|q-i\ell_0|^2}. \tag{18}$$

However, we see that the term $\ell_E\cos(kq)$ in Eq.(16) is ill–defined in the $\hbar\longrightarrow 0$ limit, a problem which has been recently considered also by Floyd 15. Thus, the existence of the classical limit implies some condition on ℓ_E. In particular, in order to reach the classical value $\sqrt{2mE}$ in the $\hbar\longrightarrow 0$ limit, the quantity ℓ_E should depend on E. In Refs. 2. and 4. it has been shown that

$$\ell_E=k^{-1}e^{-\alpha(x_p^{-1})}+e^{-\beta(x_p)}\ell_0, \tag{19}$$

where $x_p = k\lambda_p = \sqrt{2mEG/\hbar c^3}$ and

$$\alpha(x_p^{-1}) = \sum_{k\geq 1} \alpha_k x_p^{-k}, \qquad \beta(x_p) = \sum_{k\geq 1} \beta_k x_p^k. \tag{20}$$

It follows that

$$p_E = \pm \frac{2k^{-1}\hbar e^{-\alpha(x_p^{-1})} + \hbar e^{-\beta(x_p)}(\ell_0 + \bar{\ell}_0)}{2\left|k^{-1}\sin(kq) - i\left(k^{-1}e^{-\alpha(x_p^{-1})} + e^{-\beta(x_p)}\ell_0\right)\cos(kq)\right|^2}. \tag{21}$$

The function $\alpha(x_p^{-1})$ is constrained by the conditions

$$\lim_{\hbar\to 0} e^{-\alpha(x_p^{-1})} = 1, \qquad \lim_{E\to 0} E^{-1/2} e^{-\alpha(x_p^{-1})} = 0, \tag{22}$$

whereas for $\beta(x_p)$ we have

$$\lim_{\hbar\to 0} \hbar^{-1} e^{-\beta(x_p)} \ell_0 = 0. \tag{23}$$

One of the conditions we used to derive the above formulas concerns the existence of the classical limit. In this context we should observe that it may be that the classical expressions themselves may contain further terms which do not depend on $\hbar$. As an example, observe that two free particles should always contain the gravitational potential as intrinsic interaction. This seems to be connected with the residual indeterminacy discussed in 15. The aim of the present paper is to investigate the possibility that such interaction may be a consequence of the quantum potential.

The appearance of the Planck scale in the hidden constants has been considered in Refs. 2. and 4. This seems related to the 't Hooft's approach 16. Possible connections have been considered by Floyd 15. and in 5.

4 The cocycle condition and the quantum nature of interactions

Let us further consider the nature of the EP itself. We introduce the notation

$$J_{ki} = \frac{\partial q_i}{\partial q_k^v}, \tag{24}$$

and

$$(p^v|p) = \frac{\sum_k p_k^{v^2}}{\sum_k p_k^2} = \frac{p^t J^t J p}{p^t p}. \tag{25}$$

The only possibility to reach any other state $\mathcal{W}^v \neq 0$ starting from $\mathcal{W}^0$ is that it transforms with an inhomogeneous term 1.–5.

$$\mathcal{W}^v(q^v) = (p^v|p^a)\mathcal{W}^a(q^a) + (q^a; q^v), \tag{26}$$

and

$$Q^v(q^v) = (p^v|p^a)Q^a(q^a) - (q^a; q^v), \tag{27}$$

where $(q^a; q^v)$ denotes a still undefined function which depends on q^a and q^v. Let us denote by $a, b, c, \ldots$ a set of different v–transformations. Comparing

$$\mathcal{W}^b(q^b) = (p^b|p^a)\mathcal{W}^a(q^a) + (q^a; q^b) = (q^0; q^b), \tag{28}$$

with the same formula with q^a and q^b interchanged we have

$$(q^b; q^a) = -(p^a|p^b)(q^a; q^b). \tag{29}$$

More generally, comparing

$$\mathcal{W}^b(q^b) = (p^b|p^c)\mathcal{W}^c(q^c) + (q^c; q^b) = (p^b|p^a)\mathcal{W}^a(q^a) + (p^b|p^c)(q^a; q^c) + (q^c; q^b), \tag{30}$$

with (28) we obtain the basic cocycle condition

$$(q^a; q^c) = (p^c|p^b)\left[(q^a; q^b) - (q^c; q^b)\right], \tag{31}$$

which is the essence of the EP. In particular, this condition unequivocally leads to determine the correction to the CHJE. In doing this, one shows that Eq.(31) implies a basic Möbius invariance of $(q^a; q^b)$. The $\mathcal{W}^0 \equiv 0$ state plays a special role. Setting $\mathcal{W}^a = \mathcal{W}^0$ in Eq.(26) yields $\mathcal{W}^v(q^v) = (q^0; q^v)$. Thus, in general

$$\mathcal{W}(q) = (q^0; q), \tag{32}$$

so that, according to the EP (5), all states correspond to the inhomogeneous part in the transformation of the $\mathcal{W}^0$ state induced by some v–map. Since the inhomogeneous part has a purely quantum origin, we conclude that the Equivalence Postulate implies that interactions have a purely quantum origin.

The role of the quantum potential as responsible for interactions can be made clearer from the observation that the EP implies

$$\mathcal{W}^v(q^v) + Q^v(q^v) = (p^v|p)\left(\mathcal{W}(q) + Q(q)\right). \tag{33}$$

Then, taking $\mathcal{W} = \mathcal{W}^0 \equiv 0$ and omitting the superscript v, we have

$$\mathcal{W}(q) = (p|p^0)Q^0(q^0) - Q(q), \tag{34}$$

showing that any potentials can be expressed in quantum terms.

In 5. it has been observed that there is a hidden antisymmetric tensor in QM which arises from the continuity equation. We also note that in the one–dimensional case, the freedom deriving from the underlying hidden tensor one meets in the higher dimensional case reflects itself in the appearance of the integration constants. These are related to the $SL(2,\mathbb{C})$ symmetry

$$e^{2i\mathcal{S}_0/\hbar} \longrightarrow \frac{Ae^{2i\mathcal{S}_0/\hbar} + B}{Ce^{2i\mathcal{S}_0/\hbar} + D}, \tag{35}$$

of the equation

$$\{e^{2i\mathcal{S}_0/\hbar}, q\} = -4m\mathcal{W}/\hbar^2, \tag{36}$$

which is equivalent to the QSHJE (6). In particular, as we said, there is a complex integration constant ℓ which is missing in the Schrödinger equation. Changing this constant corresponds to a Möbius transformation (35). While this leaves $\mathcal{W}$ unchanged, it mixes the quantum potential and the kinetic term. Thus the quantum potential is essentially parameterized by $SL(2,\mathbb{C})$ transformations in which the constants A, B, C and D depend, by dimensional analysis and consistency of relevant limits considered above, on fundamental constants. We may expect that these constants and the above Möbius transformations in three and four dimension (for the relativistic generalization) should be related to fundamental interactions.[1]

[1]It is worth mentioning that the geometrical building block of string theory, which also explains why string

5 The two–particle model

The above investigation suggests considering that the quantum potential Q is at the origin of the interactions. Thus, it may be that the constants defining Q depend on the intrinsic properties of the particles. This would lead to different possible forms of Q and therefore of the admissible interactions. As we saw, there are subtle questions concerning the classical limit. Similarly, one should consider the relativistic case as it may lead to results which may remain hidden if considered directly in the non–relativistic case. Similarly, at least in the case of gravitational interaction, one should consider the analysis of macroscopic objects to take into account possible collective effects. Nevertheless, the above suggestions indicate that it is worth studying the case of two free particles and then looking at the possible structure of the quantum potential.

In the case of two free particles of energy E and masses m_1 and m_2, the QSHJE reads

$$\frac{1}{2m_1}(\nabla_1\mathcal{S}_0)^2+\frac{1}{2m_2}(\nabla_2\mathcal{S}_0)^2-E-\frac{\hbar^2}{2m_1}\frac{\Delta_1 R}{R}-\frac{\hbar^2}{2m_2}\frac{\Delta_2 R}{R}=0. \tag{37}$$

The continuity equation is

$$\frac{1}{m_1}\nabla_1\cdot(R^2\nabla_1\mathcal{S}_0)+\frac{1}{m_2}\nabla_2\cdot(R^2\nabla_2\mathcal{S}_0)=0. \tag{38}$$

Next, we set

$$r=r_1-r_2,\qquad r_{c.m.}=\frac{m_1r_1+m_2r_2}{m_1+m_2},\qquad m=\frac{m_1m_2}{m_1+m_2}, \tag{39}$$

where r_1 and r_2 are the ray vectors of the two particles. With respect to the new variables the equations (37) and (38) have the form

$$\frac{1}{2(m_1+m_2)}(\nabla_{r_{c.m.}}\mathcal{S}_0)^2+\frac{1}{2m}(\nabla\mathcal{S}_0)^2-E-\frac{\hbar^2}{2(m_1+m_2)}\frac{\Delta_{r_{c.m.}}R}{R}-\frac{\hbar^2}{2m}\frac{\Delta R}{R}=0, \tag{40}$$

$$\frac{1}{m_1+m_2}\nabla_{r_{c.m.}}\cdot(R^2\nabla_{r_{c.m.}}\mathcal{S}_0)+\frac{1}{m}\nabla\cdot(R^2\nabla\mathcal{S}_0)=0, \tag{41}$$

theory includes gravity, is the thrice punctured Riemann sphere. The latter can be characterized just by the basic $SL(2,\mathbb{C})$ Möbius symmetry related to the arbitrariness of the position of the punctures.

where ∇ ($\nabla_{r_{c.m.}}$) and Δ ($\Delta_{r_{c.m.}}$) are the gradient and Laplacian with respect to the components of the vector r ($r_{c.m.}$). These equations can be decomposed into the equations for the center of mass $r_{c.m.}$ and those for the relative motion. We will concentrate on the latter. It satisfies the QSHJE

$$\frac{1}{2m}(\nabla \mathcal{S}_0)^2 - E - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0, \tag{42}$$

and the continuity equation

$$\nabla \cdot (R^2 \nabla \mathcal{S}_0) = 0. \tag{43}$$

In 5 it has been stressed that the continuity equation implies

$$R^2 \partial_i \mathcal{S}_0 = \epsilon_i{}^{i_2 \ldots i_D} \partial_{i_2} F_{i_3 \ldots i_D}, \tag{44}$$

where F is a $(D-2)$–form. In the 3D case $R^2\partial_i \mathcal{S}_0$ is the curl of a vector that we denote by B

$$\nabla \mathcal{S}_0 = R^{-2} \nabla \times B. \tag{45}$$

The QSHJE (42) reduces to the "canonical form"

$$j^2 = \hbar^2 R^3 \Delta R + 2mER^4, \tag{46}$$

where $j^2 \equiv j_k j^k$ with

$$j = \nabla \times B. \tag{47}$$

Using the identity $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$, Eq.(46) reads

$$\Delta B^2 - (\nabla B)^2 = \hbar^2 R^3 \Delta R + 2mER^4. \tag{48}$$

It is worth stressing that j resembles the usual current. However, besides the mass term, we stress again that here R and $\mathcal{S}_0$ are not in general the ones one obtains identifying $Re^{\frac{i}{\hbar}\mathcal{S}_0}$ with the wave–function. Nevertheless, by construction we have that $\psi = Re^{\frac{i}{\hbar}\mathcal{S}_0}$ solves the Schrödinger equation. Thus, we have

$$j = \frac{i\hbar}{2}(\psi \nabla \overline{\psi} - \overline{\psi} \nabla \psi), \tag{49}$$

that, in the case in which ψ is the wave–function, coincides, upon dividing by m, with the usual quantum mechanical current.

We have seen that two free particles have a non–trivial quantum potential whose structure depends on the field B. In the following we will use the above results in order to investigate whether this potential may in fact have a gravitational leading behavior.

6 Gravitational interaction and quantum potential

After summarizing the main results so far, we will write down the differential equation that the quantum potential should satisfy in order to obtain the gravitational interaction. We will then investigate in some detail such an equation.

The aim of the previous sections was to show the main aspects suggesting that the quantum potential is at the origin of fundamental interactions. Even if these aspects have been discussed in detail, it is useful to collect them together before formulating the hypothesis and then deriving the relevant equations.

1. The EP implies that the reduced action is always non–trivial. In particular, this is true also for the free particle of vanishing energy. Furthermore, if $\psi \propto \overline{\psi}$, such as in the case of the wave–function for bound states, then $\psi = R\left(Ae^{-\frac{i}{\hbar}\mathcal{S}_0} + Be^{\frac{i}{\hbar}\mathcal{S}_0}\right)$, with $\psi \propto \overline{\psi}$ giving $|A| = |B|$. Thus there is no track of the condition $\mathcal{S}_0 = cnst$. On the other hand, this cannot be a solution of the QSHJE and would give an inconsistent classical limit. Remarkably, this answers the objections concerning the classical limit posed by Einstein. He just noticed that for a particle in a box the identification of the wave–function with $Re^{\frac{i}{\hbar}\mathcal{S}_0}$ gives $\mathcal{S}_0 = cnst$ and this cannot reproduce, in the $\hbar \to 0$ limit, the non–trivial $\mathcal{S}_0^{cl}$. This result has been previously derived by Floyd in a series of important papers 6. Related aspects have also been considered in the interesting papers by Reinisch[2] 9.

[2] I am grateful to G. Reinisch who informed me that the argument about the unphysical $\hbar \to 0$ limit was explicitly used by Einstein (see pg.243 of Holland's book 8.).

2. This property of the reduced action implies the existence of an intrinsic potential energy which, like the rest mass of special relativity, is universal. In particular, the quantum potential is always non–trivial. This is different from the standard approach where there are examples in which $Q = 0$ so that the QHJE would coincide with the classical one.

3. The existence of the classical limit implies that the quantum potential depends, through the hidden initial conditions coming from the QSHJE, on fundamental length scales which in turn depend on $\hbar$. It is a basic fact that these initial conditions are missing in the Schrödinger equation. In particular, the emergence of the Planck length, and therefore of Newton's constant, arises from considering the classical limit for the free particle of vanishing energy.

4. It can be seen in the formulation that the quantum potential provides particle's response to an external perturbation. For example, in the case of tunnelling, the attractive nature of the quantum potential guarantees the reality of the conjugate momentum and therefore of the velocity field $v = 1/\partial_E p \neq p/m$ (see Refs. 6. and 4.). More precisely, inside the barrier the quantum potential decreases its value in such a way that $(\partial_q \mathcal{S}_0)^2$ remains positive definite. As a consequence, the role of this internal energy, which is a property of all forms of matter, should manifest itself through effective interactions depending on the above fundamental constants.

5. The fundamental implication of the EP is the cocycle condition (31). In particular, from this condition, one obtains an expression for the interaction terms which is purely of quantum origin.

6. The fact that QM arises from an EP which is reminiscent of Einstein's EP strongly indicates a deep relation between gravitation and QM itself.

The most characteristic property of the quantum potential is its universal nature: it is a property possessed by all forms of matter. On the other hand, we know that such a property

is the one characterizing gravity. Therefore, if we write down the classical equations of motion for a pair of particles, we should always include, already at the classical level, the gravitational interaction. Furthermore, the quantum potential for a free particle is negative definite. This should be compared with the attractive nature of gravity.

7 The quantum potential with the gravitational potential as a leading term

The above remarks suggest formulating the hypothesis that the quantum potential is in fact at the origin of gravitation. Thus we look for solutions of the QSHJE leading to the classical HJ equation for the gravitational interaction. In particular, we should investigate whether in the case of two free particles the quantum potential

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R}, \tag{50}$$

admits the form

$$Q = V_G, \tag{51}$$

with V_G reducing to the Newton potential in the $\hbar \to 0$ limit

$$\lim_{\hbar\to 0} V_G = -G\frac{m_1 m_2}{r}. \tag{52}$$

If such a solution exists then, in the limit $\hbar \to 0$, Eq.(42) corresponds to the HJ equation for the gravitational potential

$$\frac{1}{2m}(\nabla \mathcal{S}_0^{cl})^2 - G\frac{m_1 m_2}{r} - E = 0. \tag{53}$$

Summarizing, the above problem corresponds to finding all the possible R satisfying the equation

$$\frac{\hbar^2}{2m}\frac{\Delta R}{R} = -V_G = G\frac{m_1 m_2}{r} + \mathcal{O}(\hbar), \tag{54}$$

where the higher order terms $\mathcal{O}(\hbar)$ will generally depend on r, such that R and $\mathcal{S}_0$ satisfy Eqs.(42) and (43). Let us consider the set $\mathcal{R} = \{R|sol.\ of\ (54)\}$. The above problem is

equivalent to find the set $\mathcal{B} = \{B|sol.\ of\ (48)\ with\ R \in \mathcal{R}\}$ (recall that if R and B solve Eq.(48), then $\nabla\mathcal{S}_0 = R^{-2}\nabla \times B$ is solution of the QSHJE and of the continuity equation). It follows that the set of possible potentials with gravitational behavior r^{-1} is given by

$$\mathcal{V}_G = \left\{ -\frac{\hbar^2}{2m}\frac{\Delta R}{R} \middle| R \in \mathcal{R}_G \right\}, \tag{55}$$

where $\mathcal{R}_G = \{R|R \in \mathcal{R}, B\ exists\}$. In other words, we have to find all the possible R satisfying (54) and then restricting to those for which there exists a field B satisfying (48). This would fix the set of admissible potentials $\mathcal{V}_G$ to be investigated. Note that the fact that the higher order terms in (54) are not fixed implies that $\mathcal{R}$ has infinitely many elements. This set identifies infinitely many equations of the kind (48), one for each $R \in \mathcal{R}$. Thus, on general grounds, one should expect that the set $\mathcal{B}$, and therefore $\mathcal{V}_G$, be non–trivial.

8 The spherical case

While an adequate treatment of the above problem will be considered in a future publication, here we consider some related preliminary aspects. By introducing the B field we saw that it should be possible to a find a solution to the two–particle model. However, a more effective way of considering such a problem seems to reformulate it as follows. First we note that by (42) and (54) we have that $\mathcal{S}_0$ should satisfy the equation

$$\frac{1}{2m}(\nabla\mathcal{S}_0)^2 = E + G\frac{m_1 m_2}{r} + \mathcal{O}(\hbar). \tag{56}$$

Thus, instead of finding first the possible $R \in \mathcal{R}$, it seems convenient to solve Eq.(56) which looks simpler than Eq.(54). A general solution of this equation would involve terms depending also on θ and ϕ. However, the simplest situation is when $\mathcal{S}_0$ is a function of r. In this case $\nabla\mathcal{S}_0 = \hat{r}\partial_r\mathcal{S}_0(r)$, where $\hat{r}$ is the unit vector along r. Eq.(56) becomes

$$\frac{1}{2m}(\partial_r\mathcal{S}_0)^2 = E + G\frac{m_1 m_2}{r} + \mathcal{O}(\hbar), \tag{57}$$

and the continuity equation reads $\nabla \cdot (R^2\hat{r}\partial_r\mathcal{S}_0) = 0$, giving

$$R = \frac{1}{r\sqrt{\partial_r\mathcal{S}_0}}. \tag{58}$$

Since the radial part of the Laplacian is $r^{-1}\partial_r^2 r$, we have that the QSHJE (42) becomes

$$\frac{1}{2m}(\partial_r\mathcal{S}_0)^2 - E + \frac{\hbar^2}{4m}\{\mathcal{S}_0, r\} = 0. \tag{59}$$

Formally this equation is the one–dimensional QSHJE for a free particle on the non–negative part of the real axis. Therefore, by Eq.(16) we have

$$\partial_r\mathcal{S}_0 = \pm\frac{\hbar(\ell_E + \bar{\ell}_E)}{2\,|k^{-1}\sin(kr) - i\ell_E\cos(kr)|^2}. \tag{60}$$

To establish the right asymptotic we should handle the indeterminacy discussed above and eliminated by a suitable choice of the constant ℓ_E. In particular, while in the previous case the structure of ℓ_E was fixed by requiring that $p_E \to \pm\sqrt{2mE}$ as $\hbar \to 0$, we should now investigate the full functional structure of the right hand side of (60) at the different scales defined by the parameters ℓ_E, $\hbar$, m and E.

We now consider the general case by adding to $\mathcal{S}_0$ the dependence on θ and ϕ and then studying the possible appearance of the r^{-1} term. Setting $\psi = Re^{\frac{i}{\hbar}\mathcal{S}_0}$, which is a solution of the Schrödinger equation, we have $\mathcal{S}_0 = \frac{\hbar}{2i}\ln(\psi/\overline{\psi})$, so that

$$(\nabla\mathcal{S}_0)^2 = -\frac{\hbar^2}{4|\psi|^4}\sum_{j=1}^{3}(\overline{\psi}\partial_j\psi - \psi\partial_j\overline{\psi})^2, \tag{61}$$

where $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ and $\partial_3 = \partial_z$. Since ψ solves the free Schrödinger equation, we have

$$\psi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{j=1}^{2} c_{lmj}R_{klj}(r)Y_{lm}(\theta,\phi), \tag{62}$$

where the $Y_{lm}(\theta,\phi)$ denote the spherical harmonics and

$$R_{kl1} = (-1)^l 2\frac{r^l}{k^l}\left(\frac{1}{r}\partial_r\right)^l\frac{\sin kr}{r}, \qquad R_{kl2} = (-1)^l 2\frac{r^l}{k^l}\left(\frac{1}{r}\partial_r\right)^l\frac{\cos kr}{r}. \tag{63}$$

These are linearly independent solutions of the radial part of the Schrödinger equation

$$R''_{klj} + \frac{2}{r}R'_{klj} + \left[k^2 - \frac{l(l+1)}{r^2}\right]R_{klj} = 0. \tag{64}$$

We are studying this equation at $r > 0$. In this respect note that the singularity of R_{kl2} at $r = 0$ would give a term $\delta(r)$ in the right hand side of (64). In spherical coordinates we have

$$(\nabla\mathcal{S}_0)^2 = -\frac{\hbar^2}{4|\psi|^4}\left[(\overline{\psi}\partial_r\psi - \psi\partial_r\overline{\psi})^2 + \frac{1}{r^2}(\overline{\psi}\partial_\theta\psi - \psi\partial_\theta\overline{\psi})^2 + \frac{1}{r^2\sin^2\theta}(\overline{\psi}\partial_\phi\psi - \psi\partial_\phi\overline{\psi})^2\right]. \tag{65}$$

The properties of this expression will be considered elsewhere. However, as a preliminary step, we consider the first term in the square bracket in (65). Note that

$$\nabla = \left(\partial_r, \frac{1}{2r}(e^{-i\phi}l_+ - e^{i\phi}l_-), \frac{i}{r\sin\theta}l_z\right), \tag{66}$$

where $l_\pm = l_x \pm il_y = e^{\pm i\phi}(\partial_\theta + i\cot\theta\partial_\phi)$, with l_x, l_y and l_z denoting the components of the angular momentum operator. Since $l_+Y_{lm} = a_{lm}Y_{lm+1}$ and $l_-Y_{lm} = a_{lm-1}Y_{lm-1}$, with $a_{lm} \equiv \sqrt{(l+m+1)(l-m)}$, we have

$$\nabla\psi = \sum_{\{lmj\}}\left(c_{lmj}R'_{klj}Y_{lm}, \frac{1}{2r}c_{lmj}R_{klj}(e^{-i\phi}a_{lm}Y_{lm+1} - e^{i\phi}a_{lm-1}Y_{lm-1}), \frac{i}{r\sin\theta}c_{lmj}R_{klj}mY_{lm}\right), \tag{67}$$

where $\sum_{\{lmj\}} \equiv \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{j=1}^{2}$. Finally, using $R'_{klj} = lr^{-1}R_{klj} - kR_{kl+1j}$, we have

$$(\overline{\psi}\partial_r\psi - \psi\partial_r\overline{\psi})^2 = -4\Im^2\left(\sum_{\{lmj\}}\sum_{\{l'm'j'\}}\overline{c}_{l'm'j'}R_{kl'j'}\overline{Y}_{l'm'}c_{lmj}(lr^{-1}R_{klj} - kR_{kl+1j})Y_{lm}\right). \tag{68}$$

9 Conclusions

Let us conclude by observing that the aim of the present investigation is to propose a possible quantum origin of the gravitational interaction. In particular, we made a preliminary investigation of the problem of finding the set $\mathcal{V}_G$ of potentials with gravitational leading term originated by the quantum potential of two free particles. The general solution seems to be of mathematical interest and will be considered in a future publication. In this context we would like to mention the Schrödinger–Newton equation 17. which concerns a problem reminiscent of the one introduced in this paper. Finally, we would like to mention that geometrical aspects related to the quantum HJ equation have been considered also in 18. and references therein.

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